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The Sharp Lipschitz Constants for Feasible and Optimal Solutions of a Perturbed Linear Program

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ABSTRACT

The purpose of this paper is to derive the sharp Lipschitz constants for the feasible solutions and optimal solutions of a linear program with respect to right-hand side perturbations. The Lipschitz constants are given in terms of pseudoinverses of submatrices of the matrices involved and are proven to be sharp.

1. INTRODUCTION

Consider the following linear programming problem:

$$\min(b, d) := \min\{c^T x : Ax \leq b, Cx = d\}, \quad (1.1)$$

where A is an $m \times n$ matrix, C a $k \times n$ matrix, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $d \in \mathbb{R}^k$. Let

$$F\begin{pmatrix} b \\ d \end{pmatrix} := \{x \in \mathbb{R}^n : Ax \leq b, Cx = d\}$$

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denote the feasible set of (1.1), and

$$S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) := \{x \in \mathbb{R}^n : Ax \leq b, Cx = d, c^T x = \min(b, d)\}$$

the solution set of (1.1). In general, F and S are set-valued mappings (or multifunctions). For simplicity, we shall always assume that all row vectors of C are linearly independent and

$$S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) \neq \emptyset \quad \text{for some } b \in \mathbb{R}^m, \quad d \in \mathbb{R}^k.$$

It was first shown by Hoffman [6] that F is Lipschitz continuous; i.e., there exists a scalar $\alpha > 0$ such that

$$H\left(F\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right), F\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)\right)_\nu \leq \alpha \cdot \left\|\begin{smallmatrix} b \\ d \end{smallmatrix}\right\|_\mu - \left\|\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right\|_\mu$$

for $b, b' \in \mathbb{R}^m, \quad d, d' \in \mathbb{R}^k, \quad (1.2)$

where $\|\cdot\|_\mu, \|\cdot\|_\nu$ denote two arbitrary norms and $H(\cdot, \cdot)_\nu$ denotes the Hausdorff metric induced by $\|\cdot\|_\nu$:

$$H(K, G)_\nu := \max \left\{ \sup_{x \in K} \inf_{y \in G} \|x - y\|_\nu, \sup_{y \in G} \inf_{x \in K} \|y - x\|_\nu \right\}$$

for $K, G \subset \mathbb{R}^n$.

The α in (1.2) is also known as Lipschitz constant for the solutions of the linear system $Ax \leq b, Cx = d$ with respect to right-hand-side perturbations (cf. [4]). It follows easily from (1.2) (cf. [8]) that S is Lipschitz continuous with respect to b and d ; i.e., there exists a scalar $\gamma > 0$ such that

$$H\left(S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right), S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)\right)_\nu \leq \gamma \cdot \left\|\begin{smallmatrix} b \\ d \end{smallmatrix}\right\|_\mu - \left\|\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right\|_\mu$$

for $b, b' \in \mathbb{R}^m, \quad d, d' \in \mathbb{R}^k. \quad (1.3)$

There are quite a few papers dedicated to estimation of α and γ [16, 12, 2, 13, 1, 9]. Mangasarian and Shiao [13] and Bergthaller and Singer [1] showed

that their estimates of α are better than the one given by Cook, Gerards, Schrijver, and Tardos [2] in the case that $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ are the supremum norm $\|\cdot\|_\infty$. No relation is known among the estimates of α given by Robinson [16], Mangasarian and Shiau, and Berghaller and Singer. It is worth mentioning that Mangasarian and Shiau's estimate of γ is independent of c . Their analysis shows that γ is more difficult to estimate than α . Recently, we proved that any Lipschitz constant for

$$\text{ext } F \begin{pmatrix} b \\ d \end{pmatrix}$$

(the set of all vertices of $F \begin{pmatrix} b \\ d \end{pmatrix}$) can be used as a Lipschitz constant for

$$F \begin{pmatrix} b \\ d \end{pmatrix} \quad \text{and} \quad S \begin{pmatrix} b \\ d \end{pmatrix}$$

if

$$\text{rank} \begin{pmatrix} A \\ C \end{pmatrix} = n,$$

which enables us to derive various Lipschitz constants for F and S in terms of the norms of pseudoinverses of submatrices of $\begin{pmatrix} A \\ C \end{pmatrix}$ [9].

While various estimates of α and γ are useful (cf. [5, 7, 10] for applications in convergence analysis of descent methods for solving linearly constrained minimization problems), the fundamental mathematical problem with respect to estimation of α and γ remains open: what is the smallest α in (1.2) or the smallest γ in (1.3) (i.e., what is the sharp estimate of α or γ)? In this paper, we give the sharp estimate of α and the sharp estimate of γ , which is independent of c . The sharp estimate of α leads to a natural definition of the condition number of the linear system $Ax \leq b$, $Cx = d$, while the sharp estimate of γ can be used as the condition number of the linear program (1.1) (cf. [12]). There are three features of this paper which make it different from the previous ones on estimation of α and γ :

- (1) no restriction on the norms $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$;
- (2) Lipschitz constants are given in terms of pseudoinverses of submatrices of $\begin{pmatrix} A \\ C \end{pmatrix}$;
- (3) the given Lipschitz constants are sharp (i.e., best possible).

The contents of this paper are organized as follows. In Section 2, we establish a dual characterization of best approximations from a convex poly-

hedral cone. Then we extend and improve Mangasarian and Shiau's result on Lipschitz constants for solutions of linear equalities and inequalities. Moreover, based on results obtained in [9], we establish an Lipschitz constant for solutions of the linear program (1.1) in terms of pseudoinverses of submatrices of $\begin{pmatrix} A \\ C \end{pmatrix}$. In Section 3, we derive the dual representation of the Lipschitz constant for F given in Section 2, which leads to an easy proof of the sharpness of the given Lipschitz constant for F . Also we prove the sharpness of the given Lipschitz constant for S , which is independent of c . In the classical p -norms, the Lipschitz constants are almost same as the norms of inverse matrices. Detailed comments on related works are given in Section 4.

To conclude this section we introduce some common notation used in the following sections. For any vector x (or matrix B) and an index set I , x_I (or B_I) denotes the vector (or matrix) consisting of components (or rows) of x (or B) whose indices are in I . Let $B_{I,0}$ be the matrix obtained by replacing rows of B whose indices are not in I with 0. x^T (or B^T) is the transpose of x (or B). $\text{rank}(B)$ denotes the rank of the matrix B . B^+ denotes the pseudo-inverse of B [15]. B is said to be of full row rank if the row vectors of B are linearly independent. For $x \in \mathbb{R}^n$, the p -norm of x is defined as $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leq p < \infty$, and the supremum norm $\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$. Let $\|\cdot\|_\nu$ and $\|\cdot\|_\mu$ be two arbitrary norms on \mathbb{R}^n and \mathbb{R}^{m+k} , respectively. $\|\cdot\|_\mu$ is said to be a monotone norm if $\|x\|_\mu \leq \|y\|_\mu$ whenever $|x_i| \leq |y_i|$ for $1 \leq i \leq m+k$. The dual norm of $\|\cdot\|_\nu$ is $\|x\|_{\nu^*} := \max\{x^T y : y \in \mathbb{R}^n, \|y\|_\nu = 1\}$, and $x^T y \leq \|y\|_\nu \cdot \|x\|_{\nu^*}$. For $x \in \mathbb{R}^n$ and $K \subset \mathbb{R}^n$, $d(x, K)_\nu := \inf\{\|x - y\|_\nu : y \in K\}$ is the distance from x to K in the norm $\|\cdot\|_\nu$. The upper Hausdorff metric $d(G, K)_\nu := \sup_{x \in G} d(x, K)_\nu$ for $G, K \subset \mathbb{R}^n$; and the Hausdorff metric $H(\cdot, \cdot)_\nu$ on subsets of \mathbb{R}^n with respect to $\|\cdot\|_\nu$ is defined as

$$H(K, G)_\nu := \max\{d(K, G)_\nu, d(G, K)_\nu\} \quad \text{for } G, K \subset \mathbb{R}^n.$$

To avoid the case when $K = \emptyset$ or $G = \emptyset$, we assume that $d(K, \emptyset)_\nu = d(\emptyset, K)_\nu = -\infty$. Therefore, (1.2) holds if

$$F\begin{pmatrix} b \\ d \end{pmatrix} \quad \text{or} \quad F\begin{pmatrix} b' \\ d' \end{pmatrix}$$

is empty. Similarly, (1.3) holds if

$$S\begin{pmatrix} b \\ d \end{pmatrix} \quad \text{or} \quad S\begin{pmatrix} b' \\ d' \end{pmatrix}$$

is empty. $K + G := \{x + y : x \in K, y \in G\}$. For an index set I , $|I|$ denotes the number of indices in I . Define

$$\mathcal{M}(A, C) := \left\{ I \subset \{1\}^m : |I| = \text{rank} \begin{pmatrix} A \\ C \end{pmatrix} - \text{rank } C, \text{rank} \begin{pmatrix} A_I \\ C \end{pmatrix} = \text{rank} \begin{pmatrix} A \\ C \end{pmatrix} \right\}; \quad (1.4)$$

i.e., $I \in \mathcal{M}(A, C)$ if and only if $\begin{pmatrix} A_I \\ C \end{pmatrix}$ is of full row rank and

$$\text{rank} \begin{pmatrix} A_I \\ C \end{pmatrix} = \text{rank} \begin{pmatrix} A \\ C \end{pmatrix}.$$

Note that we assume $\text{rank}(C) = k$. Then the Lipschitz constant for F will be given by the following formula:

$$\alpha_{\mu, \nu}(A, C) := \sup \left\{ \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu^*} : \begin{array}{l} \|A^T u + C^T v\|_{\nu^*} = 1; u \geq 0; \text{ the rows of } A \\ \text{corresponding to nonzero components of } u \\ \text{and the rows of } C \text{ are linearly independent} \end{array} \right\};$$

and the Lipschitz constant for S will be given by the following formula:

$$\gamma_{\mu, \nu}(A, C, c) := \max_{I \in \mathcal{M}(A, C)} \sup \left\{ d \left(\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u \\ v \end{pmatrix}, K \right)_{\nu} : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu} = 1 \right\}, \quad (1.5)$$

where $K := \{x \in \mathbb{R}^n : Ax \leq 0, Cx = 0, c^T x = 0\}$. By our assumption,

$$S \begin{pmatrix} b \\ d \end{pmatrix} \neq \emptyset \quad \text{for some } b \in \mathbb{R}^m, \quad d \in \mathbb{R}^k,$$

which implies that c is a linear combination of column vectors of A^T and C^T . Therefore, $Ax = 0$ and $Cx = 0$ imply $c^T x = 0$. Let $G := \{x \in \mathbb{R}^n : Ax = 0,$

$Cx = 0\}$. Then $G \subset K$. Therefore, the following upper bound of $\gamma_{\mu, \nu}(A, C, c)$ for all c will provide an Lipschitz constant for S which is independent of c :

$$\gamma_{\mu, \nu}(A, C) := \max_{I \in \mathcal{H}(A, C)} \sup \left\{ d \left(\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u \\ v \end{pmatrix}, G \right)_\nu : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\mu = 1 \right\}. \quad (1.6)$$

2. LIPSCHITZ CONSTANTS FOR FEASIBLE AND OPTIMAL SOLUTIONS OF A LINEAR PROGRAM

In this section we give Lipschitz constants for F and S , which extend and improve Mangasarian and Shiau's estimates. In order to derive the Lipschitz constant for F , we need three technical lemmas. The first one is a variation of Carathéodory's theorem, the second is an algebraic representation of the polar of a convex polyhedral cone, and the third is a characterization of best approximations from a convex polyhedral cone, which is similar to the Karush-Kuhn-Tucker conditions for a convex quadratic program. The Lipschitz constant for S is implicitly given in [9]. We shall outline how it can be derived here.

The proof of the following lemma is very similar to the proof of Carathéodory's theorem. But we cannot directly apply Carathéodory's theorem here, since it only yields that rows of A corresponding to nonzero components of \bar{u} and rows of C corresponding to nonzero components of \bar{v} are linearly independent (cf. Corollary 17.1.2 in [17]).

LEMMA 2.1. *For a given $y \in \mathbb{R}^n$, if the system $A^T u + C^T v = y$, $u \geq 0$ has a solution, then it has a solution $\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$ such that the rows of A corresponding to nonzero components of \bar{u} and rows of C are linearly independent.*

Proof. Consider index sets $I \subset \{i\}_1^k$ such that the system

$$A_I^T u_I + C^T v = y, \quad u_I > 0 \quad (2.1)$$

has a solution. Let J be such an index set with minimum cardinality; i.e., for any $I \subsetneq J$, (2.1) has no solution. Let $u_J > 0$ and v be such that

$$A_J^T u_J + C^T v = y. \quad (2.2)$$

Assume the contrary, that $\begin{pmatrix} A_j \\ C \end{pmatrix}$ is not of full row rank. Since C is of full row rank, there exists $i \in J$ such that

$$u_i A_i = \sum_{j \in J \setminus \{i\}} \lambda_j A_j + \sum_{j=1}^m \xi_j C_j. \quad (2.3)$$

Let $\epsilon = \min\{1, -u_j/\lambda_j : \lambda_j < 0\} \geq 0$. Define

$$\begin{aligned} \bar{u}_i &= (1 - \epsilon)u_i, \\ \bar{u}_j &= u_j + \epsilon\lambda_j \quad \text{for } j \in J \setminus \{i\}, \\ \bar{v}_j &= v_j + \epsilon\xi_j \quad \text{for } 1 \leq j \leq k. \end{aligned} \quad (2.4)$$

Let $I = \{j \in J : \bar{u}_j \neq 0\}$. It follows from (2.2)–(2.4) that

$$A_I^T \bar{u}_I + C^T \bar{v} = y.$$

By the definition of ϵ , I , and \bar{u}_j , we know that $\bar{u}_I > 0$ and $I \subsetneq J$. This contradicts the fact that (2.1) has no solution for any $I \subsetneq J$. Therefore $\begin{pmatrix} A_j \\ C \end{pmatrix}$ is of full row rank. This completes the proof of Lemma 2.1. \blacksquare

The next lemma gives an algebraic representation of the polar of a convex polyhedral cone. The lemma is a consequence of the combination lemma (cf. [14]).

LEMMA 2.2. *Suppose that $K = \{x \in \mathbb{R}^n : Ax \leq 0 \text{ and } Cx = 0\}$ and $y \in \mathbb{R}^n$. Then $y^T x \leq 0$ for $x \in K$ if and only if there exist $u \geq 0$ and v such that $y = A^T u + C^T v$.*

Now, based on the previous two lemmas, we can prove the following characterization of best approximations from a convex polyhedral cone.

LEMMA 2.3. *Suppose that $K = \{x \in \mathbb{R}^n : Ax \leq 0 \text{ and } Cx = 0\}$, $w \in \mathbb{R}^n$, and $x^* \in K$. Then $\|w - x^*\|_v = \min\{\|w - x\|_v : x \in K\}$ if and only if there exist $u \geq 0$, $u \in \mathbb{R}^m$, and $v \in \mathbb{R}^k$ satisfying the following conditions:*

- (1) $\|A^T u + C^T v\|_{v^*} = 1$;
- (2) $\begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A \\ C \end{pmatrix} (w - x^*) = \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A \\ C \end{pmatrix} w = \|w - x^*\|_v$;
- (3) *the rows of A corresponding to nonzero components of u and the rows of C are linearly independent.*

Proof. Sufficiency: Suppose that conditions (1)–(3) hold for some $u \geq 0$. For any $x \in K$, we have

$$\begin{aligned} \|w - x\|_\nu &\geq \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A \\ C \end{pmatrix} (w - x) \\ &= \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A \\ C \end{pmatrix} w - u^T A x \\ &\geq \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A \\ C \end{pmatrix} w \\ &= \|w - x^*\|_\nu, \end{aligned}$$

where the first inequality follows from condition (1) and $y^T z \leq \|y\|_\nu \cdot \|z\|_{\nu^*}$; the second inequality is the consequence of $Ax \leq 0$ and $u \geq 0$; and condition (2) implies the last equality.

Necessity: By the characterization of the best approximation from a closed convex cone [18], there exists $y \in \mathbb{R}^n$ such that

$$\begin{aligned} \|y\|_{\nu^*} &= 1, \\ y^T (w - x^*) &= y^T w = \|w - x^*\|_\nu, \\ y^T x &\leq 0 \quad \text{for } x \in K. \end{aligned} \tag{2.5}$$

By Lemma 2.2, the last condition in (2.5) implies that

$$y = \begin{pmatrix} A \\ C \end{pmatrix}^T \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

for some $\bar{u} \geq 0$. By Lemma 2.1, there exist $u \geq 0$ and v such that

$$y = \begin{pmatrix} A \\ C \end{pmatrix}^T \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.6}$$

and the rows of A corresponding to nonzero components of u and rows of C are linearly independent. Conditions (1) and (2) follow from (2.5) and (2.6). This completes the proof of Lemma 2.3. \blacksquare

THEOREM 2.4. For any $b, b' \in \mathbb{R}^m$ and $d, d' \in \mathbb{R}^k$,

$$H \left(F \begin{pmatrix} b \\ d \end{pmatrix}, F \begin{pmatrix} b' \\ d' \end{pmatrix} \right)_\nu \leq \alpha_{\mu, \nu}(A, C) \cdot \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_\mu.$$

Proof. Let $w \in F \begin{pmatrix} b \\ d \end{pmatrix}$. Let $z \in F \begin{pmatrix} b' \\ d' \end{pmatrix}$ be such that

$$\|w - z\|_\nu = \min \{ \|w - x\|_\nu : Ax \leq b, Cx = d' \}. \tag{2.7}$$

Let $I \subset \{i\}_1^k$ be the active index set of $Az \leq b$; i.e., $A_I z = b_I$ and $A_j z < b_j$ for $j \notin I$. Then it is easy to verify that (2.7) implies

$$\|w - z\|_\nu = \min\{\|w - z - x\|_\nu : A_I x \leq 0, Cx = 0\}, \quad (2.8)$$

since $z + \epsilon x$ is a solution of (2.7) for any solution x of (2.8) and sufficiently small $\epsilon > 0$. By Lemma 2.3, there exist $u_I \geq 0$ and v such that the rows of A_I corresponding to nonzero components of u_I and rows of C are linearly independent, $\|A_I^T u_I + C^T v\|_{\nu^*} = 1$, and

$$\begin{pmatrix} u_I \\ v \end{pmatrix}^T \begin{pmatrix} A_I \\ C \end{pmatrix} (w - z) = \|w - z\|_\nu.$$

Let u be the extension of u_I in \mathbb{R}^m such that $u_j = 0$ for $j \notin I$. Then

$$\begin{aligned} \|w - z\|_\nu &= \begin{pmatrix} u_I \\ v \end{pmatrix}^T \begin{pmatrix} A_I \\ C \end{pmatrix} (w - z) \\ &= \begin{pmatrix} u_I \\ v \end{pmatrix}^T \begin{pmatrix} A_I \\ C \end{pmatrix} w - \begin{pmatrix} u_I \\ v \end{pmatrix}^T \begin{pmatrix} A_I \\ C \end{pmatrix} z \\ &= \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A \\ C \end{pmatrix} w - \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} b \\ d \end{pmatrix} \\ &= \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A \\ C \end{pmatrix} w - \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} b \\ d \end{pmatrix} \\ &\leq \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} b' \\ d' \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} b \\ d \end{pmatrix} \\ &\leq \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu^*} \left\| \begin{pmatrix} b' - b \\ d' - d \end{pmatrix} \right\|_\mu \\ &\leq \alpha_{\mu, \nu}(A, C) \left\| \begin{pmatrix} b' - b \\ d' - d \end{pmatrix} \right\|_\mu, \end{aligned}$$

where the first equality follows from the definition of u_I and v ; the third and fourth equalities are derived from $u_j = 0$ for $j \notin I$, $A_I z = b_I$, and $Cz = d$;

$$w \in F \begin{pmatrix} b' \\ d' \end{pmatrix}$$

implies the first inequality; the second inequality is the property of the dual norm; and the last inequality follows from the definition of $\alpha_{\mu, \nu}(A, C)$ and properties of u, v . Thus,

$$d\left(F\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right), F\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)\right)_\nu \leq \alpha_{\mu, \nu}(A, C) \cdot \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_\mu.$$

By exchanging the roles of $\begin{pmatrix} b \\ d \end{pmatrix}$ and $\begin{pmatrix} b' \\ d' \end{pmatrix}$, we complete the proof of the theorem. \blacksquare

The following theorem on the Lipschitz constant for S is essentially Corollary 4.8 in [9].

THEOREM 2.5. *For any $b, b' \in \mathbb{R}^m$ and $d, d' \in \mathbb{R}^k$,*

$$\begin{aligned} H\left(S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right), S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)\right)_\nu &\leq \gamma_{\mu, \nu}(A, C, c) \cdot \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_\mu \\ &\leq \gamma_{\mu, \nu}(A, C) \cdot \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_\mu. \end{aligned}$$

Proof. It follows from Lemma 4.2 in [9] that

$$S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) = S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) + K,$$

where $K := \{x \in \mathbb{R}^n : Ax \leq 0, Cx = 0, c^T x = 0\}$. Thus, we have

$$\begin{aligned} d\left(S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right), S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)\right)_\nu &= d\left(S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right), S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right) + K\right)_\nu \\ &= \sup_{x \in S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)} \inf_{y \in S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right) + K} \|x - y\|_\nu \\ &= \sup_{x \in S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)} \inf_{w \in S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)} \inf_{z \in K} \|x - (w + z)\|_\nu \\ &= \sup_{x \in S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)} \inf_{w \in S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)} d(x - w, K)_\nu \\ &= \sup_{x \in S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)} \inf_{w \in S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)} \|x - w\|_s = d\left(S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right), S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)\right)_s, \end{aligned} \tag{2.9}$$

with the seminorm $\|x\|_s := d(x, K)_\nu$. [A seminorm $\|\cdot\|$ on \mathbb{R}^n is a function from \mathbb{R}^n to $[0, \infty)$ satisfying the following two conditions: (1) $\|x + y\|_s \leq \|x\|_s + \|y\|_s$ for $x, y \in \mathbb{R}^n$; (2) $\|\alpha x\|_s \leq |\alpha| \|x\|_s$ for $x \in \mathbb{R}^n$ and any scalar α .] Since all the proofs in [9] are valid if we replace the norm $\|\cdot\|_\nu$ there by a seminorm, Theorem 2.5 follows from (2.9) and Corollary 4.8 in [9] with $\|\cdot\|_\nu$ there replaced by $\|\cdot\|_s$. ■

3. PSEUDOINVERSE REPRESENTATIONS AND SHARPNESS OF LIPSCHITZ CONSTANTS

Notice that the Lipschitz constant $\alpha_{\mu, \nu}(A, C)$ for F seems to be very different from the Lipschitz constant $\gamma_{\mu, \nu}(A, C)$ for S . However, similar to the identity $\|B\|_{\mu, \nu} = \|B^T\|_{\nu^*, \mu^*}$ (the norm of a bounded linear operator is the same as the norm of its adjoint operator) in functional analysis (cf. Lemma 10(f) on p. 480 of [3]), we can derive a representation of $\alpha_{\mu, \nu}(A, C)$ in $\|\cdot\|_\mu$ and $\|\cdot\|_\nu$ norms. Such a representation facilitates an easy proof of the sharpness of $\alpha_{\mu, \nu}(A, C)$ as a Lipschitz constant for F . Also we shall prove the sharpness of $\gamma_{\mu, \nu}(A, C)$ as a Lipschitz constant for S which is independent of c .

Before proving the dual representation of $\alpha_{\mu, \nu}(A, C)$, we need the following identity about pseudoinverses, which is given in the proof of Theorem 5.1 in [9].

LEMMA 3.1. *Let $I \in \mathcal{M}(A, C)$, and \mathcal{J}_0 be the $(m+k) \times (m+k)$ diagonal matrix with diagonal elements $d_{ii} = 1$ for $i \in I$ or $i > m$, and $d_{ii} = 0$, otherwise. Then*

$$(A_{I,0}^T, C^T)^+ (A_{I,0}^T, C^T) = \begin{pmatrix} A_{I,0} \\ C \end{pmatrix} \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ = \mathcal{J}_0.$$

THEOREM 3.2.

$$\alpha_{\mu, \nu}(A, C) = \max_{I \in \mathcal{M}(A, C)} \sup \left\{ d \left(\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u \\ v \end{pmatrix}, K_I \right)_\nu : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\mu = 1 \right\},$$

where $K_I := \{x \in \mathbb{R}^n : A_I x \leq 0, Cx = 0\}$.

Proof. First note that

$$\alpha_{\mu, \nu}(A, C) \equiv \max_{I \in \mathcal{M}(A, C)} \sup \left\{ \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu^*} : \begin{array}{l} \|A_{I,0}^T u + C^T v\|_{\nu^*} = 1, u \geq 0, \text{ and} \\ \text{the components of } u \text{ corresponding} \\ \text{to zero rows of } A_{I,0} \text{ are zeros} \end{array} \right\}.$$

Let $I \in \mathcal{M}(A, C)$, u, v be such that $u \geq 0$, $\|A_{I,0}^T u + C^T v\|_{\nu^*} = 1$, $u_i = 0$ for $i \notin I$, and

$$\alpha_{\mu, \nu}(A, C) = \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu^*}.$$

Let $\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \in \mathbb{R}^{k+m}$ be such that

$$\left\| \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\|_{\mu} = 1 \quad \text{and} \quad \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}^T \begin{pmatrix} u \\ v \end{pmatrix} = \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu^*} = \alpha_{\mu, \nu}(A, C).$$

Note (cf. Lemma 3.1) that

$$\begin{pmatrix} A_{I,0} \\ C \end{pmatrix} \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ = \mathcal{J}_0.$$

Therefore, for any $w \in K_I$,

$$\begin{aligned} \alpha_{\mu, \nu}(A, C) &= \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}^T \mathcal{J}_0 \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \\ &= (A_{I,0}^T u + C^T v)^T \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \\ &\leq (A_{I,0}^T u + C^T v)^T \left[\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} - w \right] \\ &\leq \left\| \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} - w \right\|_{\nu}, \end{aligned}$$

which implies

$$\alpha_{\mu, \nu}(A, C) \leqslant d\left(\left(\begin{array}{c} A_{I,0} \\ C \end{array}\right)^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}, K_I\right)_\nu.$$

Hence,

$$\alpha_{\mu, \nu}(A, C) \leqslant \max_{J \in \mathcal{M}(A, C)} \sup \left\{ d\left(\left(\begin{array}{c} A_{J,0} \\ C \end{array}\right)^+ \begin{pmatrix} u \\ v \end{pmatrix}, K_J\right)_\nu : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\mu = 1 \right\}. \quad (3.1)$$

On the other hand, for any $I \in \mathcal{M}(A, C)$, there exists

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \in \mathbb{R}^{k+m}$$

such that

$$\left\| \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\|_\mu = 1$$

and

$$d\left(\left(\begin{array}{c} A_{I,0} \\ C \end{array}\right)^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}, K_I\right)_\nu = \sup \left\{ d\left(\left(\begin{array}{c} A_{I,0} \\ C \end{array}\right)^+ \begin{pmatrix} u \\ v \end{pmatrix}, K_I\right)_\nu : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\mu = 1 \right\}.$$

Let $w \in K_I$ be such that

$$d\left(\left(\begin{array}{c} A_{I,0} \\ C \end{array}\right)^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}, K_I\right)_\nu = \left\| \left(\begin{array}{c} A_{I,0} \\ C \end{array}\right)^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} - w \right\|_\nu.$$

Then it follows from Lemma 2.3 that there exists

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{k+m}$$

such that $\|A_{I,0}^T u + C^T v\|_{\nu^*} = 1$, $u \geqslant 0$, the components of u corresponding to zero rows of $A_{I,0}$ are zeros, and

$$(A_{I,0}^T u + C^T v)^T \left[\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right] = \left\| \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} - w \right\|_\nu.$$

Moreover, we have (cf. Lemma 3.1)

$$(A_{I,0}^T, C^T)^+ (A_{I,0}^T, C^T) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Therefore,

$$\begin{aligned} & \sup \left\{ d \left(\begin{pmatrix} A_{I,0}^T \\ C \end{pmatrix}^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}, K_I \right)_v : \left\| \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\|_\mu = 1, \bar{u} \geq 0 \right\} \\ &= (A_{I,0}^T u + C^T v)^T \begin{pmatrix} A_{I,0}^T \\ C \end{pmatrix}^+ \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \\ &= \left[(A_{I,0}^T, C^T)^+ (A_{I,0}^T, C^T) \begin{pmatrix} u \\ v \end{pmatrix} \right]^T \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \\ &= \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \leq \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu^*} \leq \alpha_{\mu, \nu}(A, C). \end{aligned} \quad (3.2)$$

The theorem follows from (3.1) and (3.2). ■

Similarly, one can derive the following dual representation of $\gamma_{\mu, \nu}(A, C)$.

THEOREM 3.3.

$$\gamma_{\mu, \nu}(A, C) = \sup \left\{ \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu^*} : \begin{array}{l} \|A^T u + C^T v\|_{\nu^*} = 1, \text{ and the rows of } A \\ \text{corresponding to nonzero components of } u \\ \text{and the rows of } C \text{ are linearly independent} \end{array} \right\}.$$

Now we are ready to prove the sharpness of $\alpha_{\mu, \nu}(A, C)$ and $\gamma_{\mu, \nu}(A, C)$.

THEOREM 3.4. *There exist $b, b' \in \mathbb{R}^m$ and $d, d' \in \mathbb{R}^k$ such that*

$$H \left(F \begin{pmatrix} b \\ d \end{pmatrix}, F \begin{pmatrix} b' \\ d' \end{pmatrix} \right)_v \geq \alpha_{\mu, \nu}(A, C) \cdot \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_\mu > 0.$$

Proof. By Theorem 3.2, there exist $I \in \mathcal{M}(A, C)$, $b^* \in \mathbb{R}^m$, and $d' \in \mathbb{R}^k$ such that

$$\left\| \begin{pmatrix} b^* \\ d' \end{pmatrix} \right\|_\mu = 1 \quad \text{and} \quad \alpha_{\mu, \nu}(A, C) = d \left(\begin{pmatrix} A_{I,0}^T \\ C \end{pmatrix}^+ \begin{pmatrix} b^* \\ d' \end{pmatrix}, K_I \right)_v.$$

Let

$$x := \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} b^* \\ d' \end{pmatrix},$$

$d = 0$, $b'_i = b_i^*$ for $i \in I$, $b'_i = b_i^* + |b_i^*| + |(Ax)_i| + 1$ for $i \notin I$, $b_i = 0$ for $i \in I$, and $b_i = |b_i^*| + |(Ax)_i| + 1$ for $i \notin I$. Then it is easy to verify that

$$x \in F \begin{pmatrix} b' \\ d' \end{pmatrix}$$

(by appealing to Lemma 3.1). Since

$$K_I \supset F \begin{pmatrix} b \\ d \end{pmatrix},$$

we have

$$\begin{aligned} H \left(F \begin{pmatrix} b' \\ d' \end{pmatrix}, F \begin{pmatrix} b \\ d \end{pmatrix} \right)_{\nu} &\geq d \left(x, F \begin{pmatrix} b \\ d \end{pmatrix} \right)_{\nu} \geq d(x, K_I)_{\nu} = \alpha_{\mu, \nu}(A, C) \\ &= \alpha_{\mu, \nu}(A, C) \cdot \left\| \begin{pmatrix} b^* \\ d' \end{pmatrix} \right\|_{\mu} \\ &= \alpha_{\mu, \nu}(A, C) \cdot \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_{\mu} > 0. \quad \blacksquare \end{aligned}$$

THEOREM 3.5. *There exist $c \in \mathbb{R}^n$, $b, b' \in \mathbb{R}^m$, and $d, d' \in \mathbb{R}^k$ such that*

$$\begin{aligned} H \left(S \begin{pmatrix} b \\ d \end{pmatrix}, S \begin{pmatrix} b' \\ d' \end{pmatrix} \right)_{\nu} &\geq \gamma_{\mu, \nu}(A, C) \cdot \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_{\mu} \\ &= \gamma_{\mu, \nu}(A, C, c) \cdot \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_{\mu} > 0. \end{aligned}$$

Proof. Let $I \in \mathcal{M}(A, C)$, $b^* \in \mathbb{R}^m$, and $d' \in \mathbb{R}^k$ be such that

$$\left\| \begin{pmatrix} b^* \\ d' \end{pmatrix} \right\|_{\mu} = 1$$

and

$$\begin{aligned} & d\left(\left(\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} b^* \\ d' \end{pmatrix}, G\right)_\nu \\ &= \max_{J \in \mathcal{M}(A, C)} \sup \left\{ d\left(\left(\begin{pmatrix} A_{J,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u \\ v \end{pmatrix}, G\right)_\nu : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\mu = 1 \right\}, \end{aligned}$$

where $G := \{x \in \mathbb{R}^n : Ax = 0, Cx = 0\}$. Let $u_i := -1$ for $i \in I$ and $u_i = 0$ for $i \notin I$. Let $c = A^T u$,

$$w := \left(\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} b^* \\ d' \end{pmatrix}\right),$$

$d = 0$, $b'_i = b_i^*$ for $i \in I$, $b'_i = b_i^* + |b_i^*| + |(Aw)_i| + 1$ for $i \notin I$, $b_i = 0$ for $i \in I$, and $b_i = |b_i^*| + |(Aw)_i| + 1$ for $i \notin I$. For any $z \in F\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)$, we have

$$c^T z = u^T A z \geq u^T b',$$

where the equality holds if and only if $(Az)_I = b'_I$. Since

$$(Aw)_I = b'_I \quad \text{and} \quad w \in F\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right)$$

(cf. Lemma 3.1), we have

$$w \in S\left(\begin{smallmatrix} b' \\ d' \end{smallmatrix}\right).$$

Similarly, one can verify that

$$z \in S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) \quad \text{if and only if} \quad Az \leq b, \quad (Az)_I = 0, \quad Cz = 0.$$

Let $K := \{x \in \mathbb{R}^n : Ax \leq 0, Cx = 0, c^T x = 0\} \supset G$. For $x \in K$, $Ax \leq 0$ and $c^T x = u^T Ax = 0$ imply $(Ax)_i = 0$ for $i \in I$. Since $I \in \mathcal{M}(A, C)$, $A_I x = 0$ and $Cx = 0$ are equivalent to $x \in G$. Therefore, $K = G$. Since

$$z \in S\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) \quad \text{if and only if} \quad Az \leq b, \quad (Az)_I = 0, \quad Cz = 0,$$

we have

$$S\begin{pmatrix} b \\ d \end{pmatrix} \subset G.$$

Hence,

$$\begin{aligned} H\left(S\begin{pmatrix} b' \\ d' \end{pmatrix}, S\begin{pmatrix} b \\ d \end{pmatrix}\right)_v &\geq d\left(w, S\begin{pmatrix} b \\ d \end{pmatrix}\right)_v \geq d(w, G)_v \\ &= \max_{J \in \mathcal{M}(A, C)} \sup \left\{ d\left(\begin{pmatrix} A_{J,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u \\ v \end{pmatrix}, G\right) : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\mu = 1 \right\} \\ &= \gamma_{\mu, v}(A, C) \cdot \left\| \begin{pmatrix} b^* \\ d' \end{pmatrix} \right\|_\mu \\ &= \gamma_{\mu, v}(A, C) \cdot \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_\mu > 0. \end{aligned}$$

Finally, it follows from the above inequality and Theorem 2.5 that $\gamma_{\mu, v}(A, C, c) = \gamma_{\mu, v}(A, C)$. \blacksquare

In p -norm and q -norm for $1 \leq p, q \leq \infty$, $a_{p,q}(A, C)$ and $\gamma_{p,q}(A, C)$ have simpler representations, which can be easily derived through the following two technical lemmas.

LEMMA 3.6. For $I \in \mathcal{M}(A, C)$, $u \in \mathbb{R}^m$, and $v \in \mathbb{R}^k$,

$$\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_I \\ C \end{pmatrix}^+ \begin{pmatrix} u_I \\ v \end{pmatrix}.$$

Proof. Let

$$I \in \mathcal{M}(A, C) \quad \text{and} \quad l := \text{rank} \begin{pmatrix} A_I \\ C \end{pmatrix}.$$

Let P_I be the $(m+k) \times l$ matrix defined by

$$P_I \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_I \\ v \end{pmatrix} \quad \text{for} \quad u \in \mathbb{R}^m, \quad v \in \mathbb{R}^k.$$

Then

$$\begin{pmatrix} A_I \\ C \end{pmatrix} = P_I \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}. \quad (3.3)$$

Notice that the two (linear) mappings in the following diagram are invertible on the indicated domains:

$$\mathcal{R}(A_{I,0}^T, C^T) \begin{pmatrix} A_{I,0} \\ C \end{pmatrix} \rightarrow \mathcal{R}(\mathcal{J}_0) \xrightarrow{P_I} \mathbb{R}^l, \quad (3.4)$$

where $\mathcal{R}(B)$ denotes the range of a matrix B and \mathcal{J}_0 is the $(m+k) \times (m+k)$ diagonal matrix with diagonal elements $d_{ii} = 1$ for $i \in I$ or $i > m$, and $d_{ii} = 0$ otherwise. Thus,

$$\begin{aligned} \begin{pmatrix} A_I \\ C \end{pmatrix}^+ &= \left[\begin{pmatrix} A_I \\ C \end{pmatrix} \Big|_{\mathcal{R}(A_{I,0}^T, C^T)} \right]^{-1} = \left[P_I \begin{pmatrix} A_{I,0} \\ C \end{pmatrix} \Big|_{\mathcal{R}(A_{I,0}^T, C^T)} \right]^{-1} \\ &= \left[\begin{pmatrix} A_{I,0} \\ C \end{pmatrix} \Big|_{\mathcal{R}(A_{I,0}^T, C^T)} \right]^{-1} (P_I|_{\mathcal{R}(\mathcal{J}_0)})^{-1} = \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ (P_I|_{\mathcal{R}(\mathcal{J}_0)})^{-1}, \end{aligned} \quad (3.5)$$

where the first and the last equality follow from an equivalent definition of the pseudoinverse (cf. [15]), and second and the third are the results of (3.3) and (3.4), respectively. For $u \in \mathbb{R}^m$, let $u_{I,0}$ be the vector such that $(u_{I,0})_i = u_i$ for $i \in I$ and $(u_{I,0})_i = 0$ otherwise. Then

$$\begin{aligned} \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \mathcal{J}_0 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u_{I,0} \\ v \end{pmatrix} \\ &= \begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ (P_I|_{\mathcal{R}(\mathcal{J}_0)})^{-1} \begin{pmatrix} u_I \\ v \end{pmatrix} = \begin{pmatrix} A_I \\ C \end{pmatrix}^+ \begin{pmatrix} u_I \\ v \end{pmatrix}. \quad \blacksquare \end{aligned}$$

LEMMA 3.7. *For any subset M of \mathbb{R}^n ,*

$$\begin{aligned} & \sup \left\{ d \left(\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u \\ v \end{pmatrix}, M \right)_q : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_p = 1 \right\} \\ &= \sup \left\{ d \left(\begin{pmatrix} A_I \\ C \end{pmatrix}^+ y, M \right)_q : \|y\|_p = 1 \right\}. \end{aligned}$$

Proof. It follows from Lemma 3.6 that

$$\begin{aligned} & \sup \left\{ d \left(\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+ \begin{pmatrix} u \\ v \end{pmatrix}, M \right)_q : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_p = 1 \right\} \\ &= \sup \left\{ d \left(\begin{pmatrix} A_I \\ C \end{pmatrix}^+ \begin{pmatrix} u_I \\ v \end{pmatrix}, M \right)_q : \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_p = 1 \right\} \\ &= \sup \left\{ d \left(\begin{pmatrix} A_I \\ C \end{pmatrix}^+ \begin{pmatrix} u_I \\ v \end{pmatrix}, M \right)_q : \left\| \begin{pmatrix} u_I \\ v \end{pmatrix} \right\|_p = 1 \right\} \\ &= \sup \left\{ d \left(\begin{pmatrix} A_I \\ C \end{pmatrix}^+ y, M \right)_q : \|y\|_p = 1 \right\}. \quad \blacksquare \end{aligned}$$

The following two corollaries are direct consequences of Lemma 3.7, Theorem 3.1, and Theorem 2.5.

COROLLARY 3.8.

$$\alpha_{p,q}(A, C) = \max_{I \in \mathcal{H}(A, C)} \sup \left\{ d \left(\begin{pmatrix} A_I \\ C \end{pmatrix}^+ y, K_I \right)_q : \|y\|_p = 1 \right\},$$

where $K_I := \{x \in \mathbb{R}^n : A_I x \leq 0, Cx = 0\}$.

COROLLARY 3.9.

$$\gamma_{p,q}(A, C) = \max_{I \in \mathcal{H}(A, C)} \sup \left\{ d \left(\begin{pmatrix} A_I \\ C \end{pmatrix}^+ y, G \right)_q : \|y\|_p = 1 \right\},$$

where $G := \{x \in \mathbb{R}^n : Ax = 0, Cx = 0\}$.

REMARK. If

$$\text{rank}\begin{pmatrix} A \\ C \end{pmatrix} = n,$$

then

$$\begin{pmatrix} A_I \\ C \end{pmatrix}^+ = \begin{pmatrix} A_I \\ C \end{pmatrix}^{-1} \quad \text{for } I \in \mathcal{M}(A, C).$$

In this case, $\mathcal{M}(A, C)$ has at most $m!/(n-k)!(m+k-n)!$ elements. Moreover, $G = \{0\}$ and

$$\gamma_{p,q}(A, C) = \max_{I \in \mathcal{M}(A, C)} \sup \left\{ \left\| \begin{pmatrix} A_I \\ C \end{pmatrix}^{-1} y \right\|_q : \|y\|_p = 1 \right\}.$$

4. COMMENTS ON RELATED WORK

First we would like to point out that Mangasarian and Shiau's analysis will produce the same sharp estimates of $\alpha_{\mu, \nu}(A, C)$ and $\gamma_{\mu, \nu}(A, C)$ when $\|\cdot\|_\nu$ is the supremum norm $\|\cdot\|_\infty$, if Lemma 2.1 in [13] is replaced by Lemma 2.1 in this paper. The Lipschitz constants for F and S given by Mangasarian and Shiau [13] are the following:

$$\alpha_{\mu, \nu}^*(A, C) := \sup \left\{ \begin{array}{l} \|A^T u + C_v^T\|_{\nu^*} = 1, u \geq 0, \text{ and the rows of} \\ \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu^*} : \begin{pmatrix} A \\ C \end{pmatrix} \text{ corresponding to nonzero components} \\ \text{of } \begin{pmatrix} u \\ v \end{pmatrix} \text{ are linearly independent} \end{array} \right\};$$

$$\gamma_{\mu, \nu}^*(A, C) := \sup \left\{ \begin{array}{l} \|A^T u + C^T v\|_{\nu^*} = 1, \text{ and the rows of } \begin{pmatrix} A \\ C \end{pmatrix} \\ \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mu^*} : \text{corresponding to nonzero components of} \\ \begin{pmatrix} u \\ v \end{pmatrix} \text{ are linearly independent} \end{array} \right\}.$$

It is obvious that $\alpha_{\mu, \nu}(A, C) \leq \alpha_{\mu, \nu}^*(A, C)$ and $\gamma_{\mu, \nu}(A, C) \leq \gamma_{\mu, \nu}^*(A, C)$. However, the following proposition shows that the ratios $\alpha_{\mu, \nu}^*(A, C)/\alpha_{\mu, \nu}(A, C)$ and $\gamma_{\mu, \nu}^*(A, C)/\gamma_{\mu, \nu}(A, C)$ can be as large as any positive number. Therefore, it is important to include the maximum number of linearly independent rows of C in the definition of $\alpha_{\mu, \nu}(A, C)$ and $\gamma_{\mu, \nu}(A, C)$.

PROPOSITION 4.1. *Let*

$$A_\epsilon = \begin{pmatrix} 1 & 0 \\ -1 & \epsilon \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Then

$$\lim_{\epsilon \rightarrow 0+} \frac{\alpha_{\mu, \nu}^*(A_\epsilon, C)}{\alpha_{\mu, \nu}(A_\epsilon, C)} = +\infty \quad (4.1)$$

and

$$\lim_{\epsilon \rightarrow 0+} \frac{\gamma_{\mu, \nu}^*(A_\epsilon, C)}{\gamma_{\mu, \nu}(A_\epsilon, C)} = +\infty. \quad (4.2)$$

Proof. If $u_1 u_2 = 0$, then $u_1^2 + u_2^2 + 2(\epsilon u_2)^2 \leq 2(u_1 - u_2)^2$ for $2\epsilon^2 \leq 1$. By $(t - s)^2 \leq 2(t^2 + s^2)$, we get

$$\begin{aligned} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_2 &= \sqrt{u_1^2 + u_2^2 + v^2} \leq \sqrt{u_1^2 + u_2^2 + 2(v - \epsilon u_2)^2 + 2(\epsilon u_2)^2} \\ &\leq \sqrt{2(u_1 - u_2)^2 + 2(v - \epsilon u_2)^2} = \sqrt{2}, \end{aligned} \quad (4.3)$$

provided

$$2\epsilon^2 \leq 1, \quad u_1 u_2 = 0, \quad \text{and} \quad \left\| \begin{pmatrix} u_1 - u_2 \\ \epsilon u_2 + v \end{pmatrix} \right\|_2 = 1.$$

By the definition of $\alpha_{\mu, \nu}(A_\epsilon, C)$, we have

$$\begin{aligned} \alpha_{2,2}(A_\epsilon, C) &= \sup \left\{ \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_2 : \left\| \begin{pmatrix} u_1 - u_2 \\ \epsilon u_2 + v \end{pmatrix} \right\|_2 = 1, \right. \\ &\quad \left. u_1 \geq 0, u_2 \geq 0, u_1 u_2 = 0 \right\}. \end{aligned} \quad (4.4)$$

It follows from (4.3) and (4.4) that

$$\alpha_{2,2}(A_\epsilon, C) \leq \sqrt{2}.$$

Similarly,

$$\alpha_{2,2}^*(A_\epsilon, C) = \sup \left\{ \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_2 : \left\| \begin{pmatrix} u_1 - u_2 \\ \epsilon u_2 + v \end{pmatrix} \right\|_2 = 1, \right. \\ \left. u_1 \geq 0, u_2 \geq 0, u_1 u_2 v = 0 \right\}. \quad (4.5)$$

Let $u_1 = u_2 = 1/\epsilon$ and $v = 0$. Then we have

$$\alpha_{2,2}^*(A_\epsilon, C) \geq \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_2 = \frac{\sqrt{2}}{\epsilon} \quad \text{for } \epsilon > 0. \quad (4.6)$$

Therefore,

$$\frac{\alpha_{2,2}^*(A_\epsilon, C)}{\alpha_{2,2}(A_\epsilon, C)} \geq \frac{1}{\epsilon} \quad \text{for } 0 < \epsilon \leq \frac{1}{2},$$

which implies

$$\lim_{\epsilon \rightarrow 0+} \frac{\alpha_{2,2}^*(A_\epsilon, C)}{\alpha_{2,2}(A_\epsilon, C)} = +\infty. \quad (4.7)$$

Since any two norms on a finite-dimensional space are equivalent, (4.1) is equivalent to (4.7).

Similarly, one can verify that the same arguments yield (4.2). ■

In two private letters, I. Singer showed the author that the analysis of the distance to a polyhedron in [1] can produce the sharp estimate $\alpha_{\mu,\nu}(A, C)$ when C is null. As a matter of fact, with Lemma 2.1, that analysis can also reproduce the sharp estimate $\alpha_{\mu,\nu}(A, C)$ even when C is not null. Therefore, the idea in Section 2 turns out to be known, and Lemma 2.1 is the key to obtaining a sharp estimate of $\alpha_{\mu,\nu}(A, C)$ when there are equality constraints involved. However, the dual representation and the sharpness of $\alpha_{\mu,\nu}(A, C)$ seem to be new.

Another related topic is the boundedness of $\alpha_{\mu,\nu}(A, C)$ and $\gamma_{\mu,\nu}(A, C)$ under perturbations of A and C .

Let $x \in \mathbb{R}^n$ be the vector constructed in the proof of Theorem 3.4. Then $Ax - b \leq b' - b$, which implies $(Ax - b)_+ \leq (b' - b)_+$. Here $(y)_+$ denotes a vector whose i th component is $\max\{0, y_i\}$. If $\|\cdot\|_\mu$ is a monotone norm, then

$$\begin{aligned} \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_\mu &\geq \left\| \begin{pmatrix} (b' - b)_+ \\ d' - d \end{pmatrix} \right\|_\mu \geq \left\| \begin{pmatrix} (Ax - b)_+ \\ d' - d \end{pmatrix} \right\|_\mu \\ &= \left\| \begin{pmatrix} (Ax - b)_+ \\ Cx - d \end{pmatrix} \right\|_\mu. \end{aligned}$$

Therefore, it follows from the proof of Theorem 3.4 that

$$d\left(x, F\left(\begin{pmatrix} b \\ d \end{pmatrix}\right)\right)_\nu \geq \alpha_{\mu, \nu}(A, C) \left\| \begin{pmatrix} (Ax - b)_+ \\ Cx - d \end{pmatrix} \right\|_\mu > 0. \quad (4.8)$$

On the other hand, for any $y \in \mathbb{R}^n$, let $d' = Cy$ and $b' = (Ay - b)_+ + b$. Then

$$y \in F\left(\begin{pmatrix} b' \\ d' \end{pmatrix}\right).$$

By Theorem 2.4, we have

$$\begin{aligned} d\left(y, F\left(\begin{pmatrix} b \\ d \end{pmatrix}\right)\right)_\nu &\leq d\left(F\left(\begin{pmatrix} b' \\ d' \end{pmatrix}\right), F\left(\begin{pmatrix} b \\ d \end{pmatrix}\right)\right)_\nu \\ &\leq \alpha_{\mu, \nu}(A, C) \left\| \begin{pmatrix} b - b' \\ d - d' \end{pmatrix} \right\|_\mu \\ &= \alpha_{\mu, \nu}(A, C) \left\| \begin{pmatrix} (Ay - b)_+ \\ Cy - d \end{pmatrix} \right\|_\mu. \end{aligned} \quad (4.9)$$

Therefore, by (4.8) and (4.9), we get

$$\alpha_{\mu, \nu}(A, C) = \max_{b \in \mathbb{R}^m, d \in \mathbb{R}^k} \max_{y \notin F\left(\begin{pmatrix} b \\ d \end{pmatrix}\right)} \frac{d\left(y, F\left(\begin{pmatrix} b \\ d \end{pmatrix}\right)\right)_\nu}{\left\| \begin{pmatrix} (Ay - b)_+ \\ Cy - d \end{pmatrix} \right\|_\mu}, \quad (4.10)$$

under the assumption that $\|\cdot\|_\mu$ is a monotone norm.

Recently, Luo and Tseng [11] investigated characterizations of boundedness of so-called Hoffman condition numbers

$$\tau(A, C, b, d) := \max_{y \in F\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)} \frac{d\left(y, F\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)\right)}{\left\|\begin{pmatrix} (Ay - b)_+ \\ Cy - d \end{pmatrix}\right\|_2}$$

under various perturbations of A, C, b, d . One case is the boundedness of $\tau(A, C, b, d)$ under local feasible perturbations of A, C and global feasible perturbations of b, d ; i.e., whether there exist $\beta > 0$ and $\epsilon > 0$ such that

$$\tau(A', C', b', d') \leq \beta \quad (4.11)$$

whenever $\|A - A'\|_2 \leq \epsilon$, $\|C - C'\|_2 \leq \epsilon$, and $A'x \leq b'$, $C'x = d'$ have a solution. It follows from (4.10) that (4.11) is equivalent to

$$\alpha_{2,2}(A', C') \leq \beta$$

whenever $\|A - A'\|_2 \leq \epsilon$ and $\|C - C'\|_2 \leq \epsilon$. Note that the particular norm used here is immaterial, since any two norms in a finite-dimensional space are equivalent. Let $\|x\|_I := d(x, K_I)_2$. The $\|\cdot\|_I$ is a seminorm. Thus, in terms of our representation of $\alpha_{2,2}(A, C)$, (4.11) is equivalent to boundedness of seminorms of pseudoinverses of certain submatrices of $\begin{pmatrix} A \\ C \end{pmatrix}$ under perturbations.

In summary, we derive the sharp Lipschitz constant for the feasible solutions F and the sharp Lipschitz constant (which is independent of c) for the optimal solutions S of a linear program. The Lipschitz constants are given in terms of seminorms of pseudoinverse $\begin{pmatrix} A_{I,0} \\ C \end{pmatrix}^+$. When

$$\text{rank}\begin{pmatrix} A \\ C \end{pmatrix} = n$$

and p -norms are involved, the Lipschitz constants are given in terms of seminorms of the inverses $\begin{pmatrix} A_I \\ C \end{pmatrix}^-$. In the 2-norm, one could get an estimate of the Lipschitz constants by computing the smallest eigenvalues of at most $m!(n-k)!(m+k-n)!$ square matrices.

One remaining open problem is whether $\gamma_{\mu,\nu}(A, C, c)$ is the sharp Lipschitz constant for S or not. Also it would be desirable to have an

algorithm which could simultaneously compute the smallest eigenvalues of all nonsingular square submatrices of the form

$$\begin{pmatrix} A_I \\ C \end{pmatrix} \begin{pmatrix} A_I \\ C \end{pmatrix}^T.$$

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